Matrix-Limited Fatigue Properties in Fibre Composite Materials

T. H. COURTNEY, J. WULFF
Department of Metallurgy, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA

Received 21 July 1966

Cyclic stressing of fibre composites in the elastic-plastic region is discussed under the assumption of equal strains in the fibre and the matrix. Under conditions of alternating load, it is found that the strain decreases and the stress on the matrix increases with increasing number of cycles. Such a phenomenon appears to be a macroscopic version of the Orowan model of fatigue.

Within certain stress-levels, it appears that the matrix may work-harden to its failure stress and the composite will fail by this mechanism. Equations are developed which relate the number of cycles to matrix failure to the composite stress. A characteristic of the equation is the existence of a matrix "endurance limit", a stress below which the matrix will not fall by progressive work-hardening, but rather by conventional fatigue mechanisms.

The analysis is applied to Ag-steel and Cu-W composites. It shows that, for stresses in excess of the "endurance limit", the number of cycles to induce failure is rather limited. At high volume-fraction of fibres in these systems, the "endurance limit" is only a small fraction of the composite tensile strength.

1. Introduction

The desire to utilise the strength of materials which, because of their brittleness or small size, cannot be used in bulk form has led in recent years to extensive work in the development and study of fibre composite materials. Such systems consist of a strong material in the form of fibre (whisker, wire, filament, or flake) embedded in a relatively soft matrix. In such a composite, optimum mechanical properties are obtained when the fibres are all aligned in the direction of the applied stress [1]. The micromechanics of such a composite are quite complicated [2-4]. However, to a first approximation, the stress-strain curve of a fibre composite* can be synthesised from the stress-strain curves of the components. This is accomplished by using the volume-fraction rule [5], where the stress on the composite, σ_c, at a given value of strain, ϵ, is obtained by a linear averaging of the stresses of the individual components (taken from their respective stress-strain curves) at the strain ϵ. It follows that

$$\sigma_c(\epsilon) = V_f\sigma_f(\epsilon) + V_m\sigma_m(\epsilon) \quad (1)$$

The subscripts, c, f, and m, stand for composite, fibre, and matrix, respectively, and V is the volume fraction. This relation holds quite well for continuous fibres and requires a slight modification [1] for discontinuous fibres. The major assumptions of the volume-fraction rule are equal strains in both the fibre and the matrix, and non-interacting stress fields between the individual components.

According to the volume-fraction rule, deformation of a fibre composite can be divided, in general, into four successive stages [5].

Stage 1 – Elastic deformation of both fibre and matrix.
Stage 2 – Elastic deformation of fibre; plastic deformation of matrix.
Stage 3 – Plastic deformation of both fibre and matrix.
Stage 4 – Failure.

The presence or absence of an individual stage

*Referring now, and for the rest of this paper, to the case when the fibres are aligned in the direction of the applied stress.

depends on the properties of the individual components. Of the four stages of deformation, the effective use of a fibre composite material depends on its properties during the second stage. In other words, when the matrix reaches its elastic limit, the stress on the composite will usually be much lower than the desired service strength of the composite. Since stage 2 more often represents the region of desired service strength, composites will most likely be designed for applications in this stress region.

As with most structural materials, potential applications of composites often involve some fluctuations in load during the life of a composite. This aspect of their behaviour has, nevertheless, received relatively little attention. Williams and O'Brien [6] and Forsyth *et al* [7] have studied fatigue properties of steel fibre-reinforced Al and obtained positive results. One of the more extensive studies was performed on silica-reinforced Al by Baker and Cratchley [8]. They have pointed out that the behaviour of fibre-reinforced materials under cyclic loading is considerably more complex than ordinary fatigue behaviour. In the present paper, some of the problem areas to be expected during a fluctuating load on a composite, corresponding to loads of stage 2 deformation, are considered. Moreover, a potentially serious limitation to the behaviour of a fibre composite under an alternating load needs to be emphasised. This limitation may be called "matrix-limited fatigue failure". It is, nevertheless, necessary to point out that any failure which takes place during cyclic loading of a composite may not be a fatigue failure in the usual sense, albeit some ordinary fatigue damage may be occurring simultaneously. In this regard, it is necessary to point out that we are only concerned in this present paper with one aspect of the behaviour of composite materials during cyclic loading, namely the work-hardening of the matrix during successive cycles. The constraint of equal deformation in the fibre and the matrix leads naturally to a successively higher load on the matrix. This is why the geometrical model of fatigue, first used by Orowan [9] and shown in fig. 1, appears applicable to composites. That it may not be in accord with the results of fatigue studies on ordinary engineering materials need not concern us in the special case described below.

2. Cyclic Deformation

In order to show how cyclic loading in stage 2 of a fibrous composite material can lead to failure of the softer matrix, consider the behaviour of the material during a half-cycle of stress. Let the i-$\frac{1}{2}$ cycle correspond to negative stress and the i th cycle to positive stress. Also, let the stress on the composite at the end of stage 1 deformation be denoted as $\sigma_{ce} = V_f E_f \epsilon_e + V_m E_m \epsilon_e$ where the E's refer to respective elastic moduli and ϵ_e to the elastic limit of

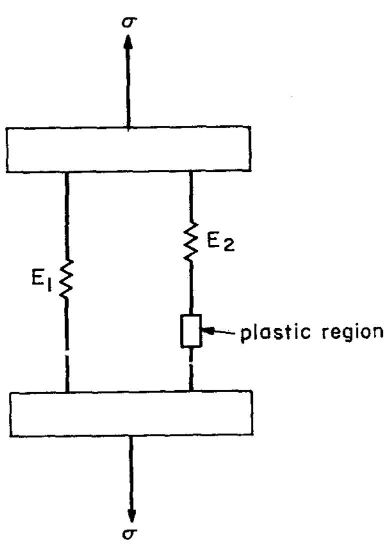


Figure 1 Mechanical model corresponding to the Orowan theory of fatigue.

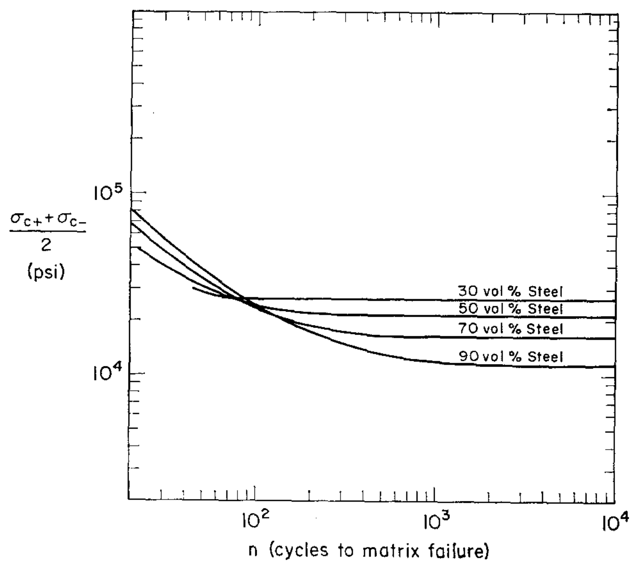


Figure 2 Relation between composite stress and cycles to matrix failure (equation 11) for Ag-steel fibre composites.

strain in the matrix. If we neglect the Bauschinger effect [10], the magnitude of σ_{ce} in tension will be the same as in compression. Hence, rewriting equation 1 in magnitude form gives, for the i-$\frac{1}{2}$ cycle,

$$\sigma_{c-} - \sigma_{ce} = V_f E_f \epsilon_p(i\text{-}\tfrac{1}{2}) + V_m \sigma_m(i\text{-}\tfrac{1}{2}) \quad (2)$$

and for the i th cycle

$$\sigma_{c+} - \sigma_{ce} = V_f E_f \epsilon_p(i) + V_m \sigma_m(i) \quad (3)$$

where σ_{c-} is the magnitude of the stress on the composite in compression and σ_{c+} is the corresponding quantity in tension, ϵ_p is the plastic strain (total strain $-\epsilon_e$) in the matrix at the stated half-cycle, and σ_m is the stress in the matrix in excess of the yield stress at the same half-cycle. Adding equations 2 and 3 gives

$$\sigma_{c+} + \sigma_{c-} - 2\sigma_{ce} = V_f E_f\{\epsilon_p(i) + \epsilon_p(i\text{-}\tfrac{1}{2})\} + V_m\{\sigma_m(i) + \sigma_m(i\text{-}\tfrac{1}{2})\} \quad (4)$$

For small changes in σ_m, we have for the change in magnitude of σ_m due to work-hardening of the matrix,

$$\sigma_m(i) - \sigma_m(i\text{-}\tfrac{1}{2}) = \sigma^*\{\epsilon_p(i) + \epsilon_p(i\text{-}\tfrac{1}{2})\} \quad (5)$$

where $\sigma^* = d\sigma/d\epsilon$, the strain-hardening characteristic of the matrix. By combining equations 4 and 5, the increase in stress on the matrix in a half-cycle becomes

$$\Delta\sigma_m(\text{half-cycle}) = \sigma_m(i) - \sigma_m(i\text{-}\tfrac{1}{2}) =$$

$$\frac{2\sigma^* V_m}{V_f E_f + V_m \sigma^*}\left\{\frac{\frac{\sigma_{c+} + \sigma_{c-}}{2} - \sigma_{ce}}{V_m} - \sigma_m(i\text{-}\tfrac{1}{2})\right\} \quad (6)$$

Using the definition of average stress,

$$\sigma_{avg} = \sigma_m(i\text{-}\tfrac{1}{2}) + \frac{\Delta\sigma}{2} \quad (7)$$

and rearranging (6) gives

$$\Delta\sigma_m(\text{half-cycle}) = A(B - \sigma_{avg}) \quad (8)$$

where

$$A = \frac{2\sigma^* V_m}{V_f E_f} \text{ and } B = \frac{\frac{\sigma_{c+} + \sigma_{c-}}{2} - \sigma_{ce}}{V_m}$$

If the stress change over the half-cycle is small, the above equation can be transformed into a differential equation relating the change in stress, $d\sigma$, to the number of half-cycles, dk, so that

$$\frac{d\sigma}{A(B - \sigma)} = dk \quad (9)$$

The number of cycles necessary for matrix failure, n, may be obtained by integrating dk from 0 to $2n$ (counting the initial loading as a quarter of a cycle) and the left-hand side of equation 9 between $\sigma = 0$ and $\sigma = \sigma_f - \sigma_y$, where σ_f = failure stress and σ_y = yield stress of the matrix.

Although it is not obvious from the above derivation, the same form of equation is obtained when the composite is cycled between two positive loads in stage 2 deformation, except that B is replaced by B' where

$$B' = \frac{\frac{\sigma_{c+} - \sigma_{c-}}{2} - \sigma_{cem}}{V_m} \quad (10)$$

with $\sigma_{cem} = E_m V_m \epsilon_e$ and $\sigma_{c+} - \sigma_{c-}$ is the range of stress.

Two solutions to equation 9 are given below.

2.1. Solution for Constant Strain-Hardening

Under the condition of constant σ^*, the relation between the number of full cycles to matrix failure, n, and the composite stress is given by

$$n = \frac{1}{2A}\ln\left\{\frac{B}{B - (\sigma_f - \sigma_y)}\right\} \quad (11)$$

2.2. Solution for $\sigma_m = K\epsilon_p{}^{\alpha}$

Equation 11, since it assumes linear work-hardening and hence, in general, over-estimates the additional stress per cycle, will under-value the cycles to failure. Using the better approximation for plastic deformation, $\sigma_m = K\epsilon_p{}^{\alpha}$, (9) reduces to

$$\frac{V_f E_f \sigma^{\frac{(1-\alpha)}{\alpha}} d\sigma}{2V_m \alpha K^{1/\alpha}(B - \sigma)} = dk \quad (12)$$

when $\alpha = \frac{1}{2}$, (12) integrates to

$$n = \frac{V_f E_f}{2V_m K^2}\left[B \ln \frac{B}{B - (\sigma_f - \sigma_y)} - (\sigma_f - \sigma_y)\right] \quad (13)$$

It is necessary to point out that the Bauschinger effect will probably lead to a larger value of n, since the yield stress in compression will be less than in tension (if the initial loading is tension). To a first approximation, however, the increased initial work-hardening following the initially lower yield [10] will tend to cancel out this effect.

TABLE I Data used in composite stress-cycles-to-matrix-failure relationship (10^6 lb/in.^{2} = 700 kg/mm^2).

System	E_f (10^6 lb/in.^{2})	E_m (10^6 lb/in.^{2})	ϵ_e (x 10^4)	σ_y (10^3 lb/in.^{2})	σ_f (10^3 lb/in.^{2})	α	K (10^3 lb/in.^{2})	σ^* (10^5 lb/in.^{2})
Cu–W	50	18	3	3	45	—	—	1.0
Cu–W	50	18	3	3	45	0.5	46	—
Ag-steel	30	11	3	4	35	—	—	1.5

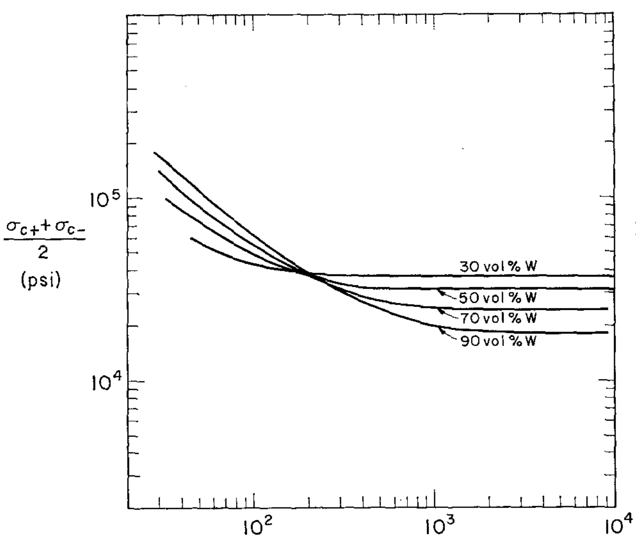
3. Applications

The results of the previous section can be applied to some composites in which the volume-fraction rule has been shown to be a good approximation [5, 11]. The materials considered are a Cu–W fibre composite and a Ag-steel fibre composite. The data used in calculating the composites' stress-cycles-to-matrix-failure relationships are shown in table I. The values of σ^* used appear to be reasonable averages for the average strain-hardening in these matrices [11, 12], as are the values of α and K used for Cu [12]. The values of σ_f used may be somewhat low. The elastic limit of strain used in the calculations is a good approximation for Cu [5], but is merely assumed for Ag. The modulus value used for W is the static modulus [5].

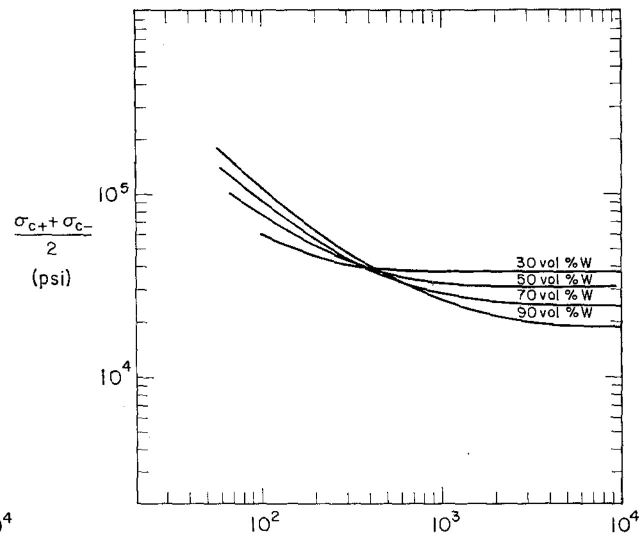
Using the values listed in table I, the calculated (equation 11), composite stress cycles to matrix failure for Ag-steel are shown in fig. 2 for various volume fractions of steel fibres. The maximum stress shown is the maximum composite stress at which the fibre still remains elastic. The results for Cu–W composites are shown in figs. 3a and b. The former is calculated from equation 11 and the latter from equation 13. Fig. 3c compares the results calculated from equation 11 with those calculated from equation 13 for two different volume-fractions. Equation 13 predicts a longer life at a given value of stress, but the "endurance limit" remains unchanged. The "endurance limit", Σ, in both cases is given by

$$\Sigma = \sigma_{ce} + V_m(\sigma_f - \sigma_y) \qquad (14)$$

According to equation 9, the stress increase per half-cycle must be small in order to transform the difference equation 6 to the differential equation 9. However, for large values of the composite stress, the stress increase per half-cycle is relatively large. In order to check this assumption, the number of cycles to failure was



(a)



(b)

Figure 3 Relation between composite stress and cycles to matrix failure for Cu–W fibre composites: (a) calculated from equation 11; (b) calculated from equation 13.

TABLE II Comparison of cycles-to-matrix-failure, as calculated from equations 6 and 11 (10^3 lb/in.^{2} = 0.7 kg/mm^2).

System	V_f	$\frac{\sigma_{c+} + \sigma_{c-}}{2}$ (10^3 lb/in.^{2})	n (equation 6)	n (equation 11)
Cu–W	0.90	180	28.5	28.8
Cu–W	0.90	140	37.7	38.1
Cu–W	0.90	100	55.8	56.3
Cu–W	0.50	100	33.3	33.3
Cu–W	0.50	80	44.8	44.7
Cu–W	0.50	60	68.6	68.5

calculated for several cases using the difference equation 6 and compared with those calculated from equation 9. The results are listed in table II. The error introduced by using the differential form of equation 6 is very small.

4. Discussion

The most unexpected result from the above is the surprisingly low number of cycles necessary for matrix failure at stresses well below the composite tensile strength. Another somewhat unexpected result is that the "endurance limit" of a fibre composite should increase as the volume fraction of the strengthening fibre decreases. Fig. 4 shows, for Cu–W and Ag-steel composites*, the relation between the volume fraction of the matrix and the ratio of the matrix-limited "endurance strength" to the composite tensile strength. For a high volume-fraction of fibre, the matrix-limited endurance strength is only a small fraction of the tensile strength. For composite stress cycles less than the endurance limit, the failure of composites under cyclic loading will probably be governed by the more usually encountered fatigue failure mechanisms. In fact, the observed stress-number of cycles-to-failure relationship will probably be an envelope of the curves of failure by work-hardening and failure by normal fatigue processes.

If the above analysis is even approximately indicative of the behaviour of real composites (and it should be in cases where the assumption of equal strains is tenable), the use of fibre composites in fatigue applications may be limited. Although the failure of the matrix will not necessarily lead to immediate failure of the composite

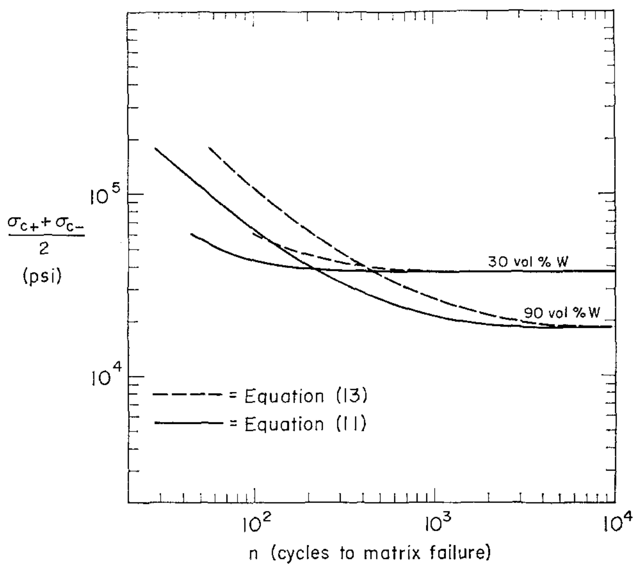


Figure 3c Comparison of equations 11 and 13 for the relation between composite stress and cycles to failure for Cu–W composites.

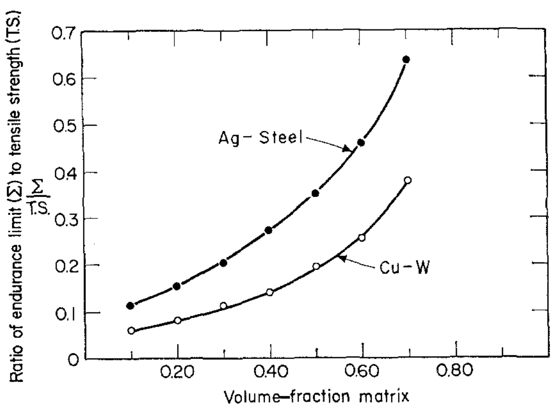


Figure 4 Matrix-limited endurance limit to tensile strength ratio as a function of volume-fraction matrix (Cu–W fibre and Ag-steel fibre composites).

*Tensile strengths taken from references 5 and 8.

(which, of course, depends on the volume fraction and tensile strength of the fibres), such matrix failure will have to be considered in design.

There are several objections that can be made to the above treatment. In addition to the Bauschinger effect previously mentioned, there is the effect of the difference in Poisson ratio between the matrix and the fibre. If one considers a plastic matrix to have a Poisson ratio of $\frac{1}{2}$ and the fibre to have an elastic Poisson ratio of less than $\frac{1}{2}$, the effective volume-fraction of fibre will be greater under load than at zero load. This will lead to the matrix carrying a smaller fraction of the applied load than assumed in this analysis and a concurrent increase in the number of cycles to failure for the matrix. This effect, however, should be fairly small. A more pertinent criticism would be the absence of a micromechanical approach to this problem. For example, the differing Poisson's ratio of the matrix and the fibre will create a non-uniaxial stress state [4] in both the fibre and the matrix, and an analysis based on these stresses may lead to different values of the cycles necessary for matrix failure. While these criticisms are all valid, the possibility still remains that cycling of a fibre composite between stresses in the elastic-plastic region will cause progressive work-hardening of the matrix which may result in its fracture. Such a mechanism will limit the use of fibre composites in applications where cyclic stresses are involved.

5. Summary

The Orowan model of fatigue has been applied to fibre composites under loads corresponding to elastic-plastic deformation. It predicts that the fatigue properties of fibre composites may be limited by the failure of the matrix caused by progressive work-hardening of the matrix. Specifically, the analysis predicts a composite stress-number of cycles-to-matrix-failure relationship, one characteristic of which is an "endurance limit". The latter is found, for high volume-fractions of the reinforcing fibres, to be only a small fraction of the composite tensile strength. Application of the equations developed to Cu–W and Ag-steel fibre composites indicates a rather limited number of cycles to failure for stresses in excess of the "endurance limit".

Acknowledgements

Professors D. H. Avery and J. W. Mar provided valuable discussion. This research was supported by contract AF 49(638)1463.

References

1. A. J. KELLY and G. J. DAVIES, *Met. Rev.* **10,** No. 37 (1965).
2. B. WALTER ROSEN, "Fibre Composite Materials" (American Society for Metals, 1965), p. 37.
3. L. J. EBERT and J. D. GADD, *ibid,* p. 89.
4. H. R. PIEHLER, AFOSR 65-1246, ASRL TR 132-1 (June 1965).
5. D. L. MCDANIELS, R. W. JECH, and J. W. WEETON, *Trans. AIME* **233** (1965) 636.
6. R. V. WILLIAMS and D. J. O'BRIEN, *Appl. Matls. Res.* (July 1964) 148.
7. P. J. E. FORSYTH, R. W. GEORGEANA, and P. A. RYDER, *ibid* (October 1964) 223.
8. A. A. BAKER and D. CRATCHLEY, *ibid,* 215.
9. E. OROWAN, *Proc. Roy. Soc. (London)* **171A** (1939) 79.
10. D. MCLEAN, "Mechanical Properties of Metals" (Wiley, New York, 1962), p. 140.
11. H. R. PIEHLER, *Trans. AIME* **233** (1965) 12.
12. J. R. LOW, "Properties of Metals in Materials Engineering" (American Society for Metals, 1949). As quoted in G. E. DIETER, "Mechanical Metallurgy" (McGraw Hill, New York, 1961), p. 248.